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Journal of Functional Analysis 232 (2006) 137–147

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Nonnegative functions as squares or sums of squares ☆

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Received 15 March 2005; accepted 22 June 2005

Communicated by G. Pisier

Available online 26 September 2005

Abstract

We prove that, for $n \geqslant 4$, there are C^∞ nonnegative functions f of n variables (and even flat ones for $n \geqslant 5$) which are not a finite sum of squares of C^2 functions. For $n = 1$, where a decomposition in a sum of two squares is always possible, we investigate the possibility of writing $f = g^2$. We prove that, in general, one cannot require a better regularity than $g \in C^1$. Assuming that f vanishes at all its local minima, we prove that it is possible to get $g \in C^2$ but that one cannot require any additional regularity.

Keywords: Sums of squares; Square roots; Nonnegative functions; Modulus of continuity; Nondifferentiability

0. Introduction

In [5], while proving their celebrated inequality, Fefferman and Phong state (and sketchily prove) a lemma assuring that any nonnegative C^∞ (indeed, $C^{3,1}$) function in

☆ This research was developed within the INdAM Project “Geometria delle classi di funzioni ultradifferenziabili ed operatori differenziali”.

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0022-1236/$ - see front matter
doi:10.1016/j.jfa.2005.06.011

$\mathbb{R}^n$ is a sum of squares of $C^{1,1}$ functions. Here $C^{k,1}$ is the space of functions whose partial derivatives up to order k are Lipschitz continuous.

In Section 1 we prove that, for $n \geqslant 4$, such a regularity condition is sharp: there exist nonnegative C^∞ functions $f : \mathbb{R}^n \to \mathbb{R}$ that are not sums of squares of C^2 functions. The core of the proof is the result of Hilbert [8] asserting that there are homogeneous polynomials of degree 4 that are not sums of squares of polynomials. For analogous reasons, there exist C^∞ nonnegative functions $\mathbb{R}^3 \to \mathbb{R}$ that are not the sum of squares of C^3 functions. Even for flat functions, which are apparently far from the polynomial situation, similar negative results occur (see Theorem 1.2).

In dimensions 1 and 2 there are no algebraic obstacles to the decomposition in sum of squares. In dimension 2, any flat nonnegative C^4 function is a sum of squares of C^2 functions; in the one-dimensional case, any C^{2m} nonnegative function is a sum of the squares of two C^m functions (see [3]).

What remains to study in dimension 1 is the case of just one square: the regularity of the square root of a nonnegative function or, more generally, the existence of a function g of a certain regularity satisfying $g^2 = f$ (we will say that g is an *admissible square root* of *f*). This is the object of Sections 2 and 3.

The starting point can be taken from the article by Glaeser [6], who proves that if $f \in C^2$ is nonnegative and 2-flat on its zeros (i.e., $f(x) = 0$ implies $f''(x) = 0$) then $f^{1/2}$ is C^1. Moreover, dropping the assumption of flatness (see [10]) one has that if f is C^2, f has an admissible square root in $C^1(\mathbb{R})$.

We prove (Theorem 2.1) that this result is sharp: given any modulus of continuity ω there are nonnegative C^∞ functions such that the first derivative of any of their admissible square roots is not ω-continuous. The case $\omega(t) = t$, i.e. that of $C^{1,1}$ admissible square roots, was already proved by Glaeser in [6].

In Section 3 we treat the case of functions whose values at all the local minima are zero or above a bound depending on the point and on the function itself. This proves to be a necessary and sufficient condition for admissible square roots to be chosen of class C^2 if starting from a C^4 function (see Theorems 3.1 and 3.5). We prove also that this result is sharp: given any modulus of continuity ω there are nonnegative C^∞ functions with value 0 at all their local minima such that the second derivative of any of their admissible square roots is not ω-continuous.

The results of Sections 2 and 3 could be therefore summarized as follows: a general nonnegative C^2 function of one variable has a C^1 admissible square root, but no better regularity can be assured; if the function is C^4 and its values at all its local minima are controlled it has a C^2 admissible square root, but no better regularity can be assured. In both cases, increasing the regularity of the nonnegative function up to C^∞ does not provide a better result.

1. Nonnegative functions as sums of squares

We recall the following theorem:

Theorem 1.1 (*Fefferman–Phong [5], Guan [7]*). *Let Ω be an open set in $\mathbb{R}^n$; then any nonnegative function f in $C^{3,1}_{loc}(\Omega)$ is a sum of squares of functions belonging to $C^{1,1}_{loc}(\Omega)$.*

Actually, the original statement of [7, Lemma 4] requires the global assumption $f \in C^{3,1}(\mathbb{R}^n)$, but the statement above is an easy consequence, thanks to the following observation (see [3, Lemma 2.1]): for any $f \in C^k_{loc}(\Omega)$ (or e.g. $C^{k,\alpha}_{loc}(\Omega)$) defined on an open subset $\Omega \subset \mathbb{R}^n$ there exist a strictly positive function $\varphi \in C^\infty(\Omega)$ and a function $g \in C^k(\mathbb{R}^n)$ (resp. e.g. $C^{k,\alpha}(\mathbb{R}^n)$) with support in $\bar{\Omega}$ such that $f = \varphi^2 g$ on Ω.

A *modulus of continuity* is a continuous increasing concave function ω, defined on an interval $[0, t_0]$, satisfying $\omega(0) = 0$. If Ω is an open subset of $\mathbb{R}^d$, a function $f : \Omega \to \mathbb{R}$ will be called *ω-continuous* on Ω if the following quantity

$$[f]_\omega = \sup_{0<|x-y|<\min(t_0, d(x,\complement\Omega)/2)} \frac{|f(y) - f(x)|}{\omega(|y-x|)}$$

is finite. For $k \in \mathbb{N}$ we will say that f belongs to $C^{k,\omega}(\Omega)$ if it belongs to C^k and if the following quantity

$$\|f\|_{k,\omega} = \|f\|_{C^k} + \sum_{|\alpha|=k} \left[\partial^\alpha f/\partial x^\alpha\right]_\omega$$

is finite. We observe that for every continuous function f on a compact set there exists a modulus of continuity ω such that f is ω-continuous and that we can always assume (as we will) that $\omega(s) \geqslant s$.

Theorem 1.2. *Let ω be a modulus of continuity. Let us consider nonnegative C^∞ functions f on R^n and possible decompositions*

$$f = \sum_{i=1}^{N} \varphi_i^2 \quad \textit{in a neighbourhood of } 0. \tag{1.1}$$

(a) *For $n \geqslant 3$, there exists f such that* (1.1) *is impossible with $\varphi_i \in C^3$;*
(b) *for $n \geqslant 4$, there exists f, flat at all its zeroes, such that* (1.1) *is impossible with $\varphi_i \in C^{3,\omega}$;*
(c) *for $n \geqslant 4$, there exists f such that* (1.1) *is impossible with $\varphi_i \in C^2$;*
(d) *for $n \geqslant 5$, there exists f, flat at all its zeroes, such that* (1.1) *is impossible with $\varphi_i \in C^{2,\omega}$.*

Proof. (a),(c) The homogeneous polynomials (see [11,4,2])

$$M(x, y, z) = z^6 + x^2 y^2 (x^2 + y^2 - 3\lambda z^2) \quad \text{in } \mathbb{R}^3 \quad \text{and}$$

$$L(x, y, z, w) = w^4 + x^2 y^2 + y^2 z^2 + z^2 x^2 - 4\lambda xyzw \quad \text{in } \mathbb{R}^4$$

are nonnegative for $0 \leqslant \lambda \leqslant 1$, vanish only at the origin for $0 < \lambda < 1$ and are not sums of squares of polynomials for $0 < \lambda \leqslant 1$.

If $p \in \mathbb{R}[x_1, \ldots, x_n]$ is a nonnegative homogeneous polynomial of degree $2d$ that is not a sum of squares of polynomials, it cannot be written as a sum of squares of C^d

functions φ_i. Otherwise, the Taylor expansion of φ_i would reduce to $\varphi_i = q_i + \mathrm{o}(|x|^d)$, with q_i homogeneous of degree d, and one would have $\sum_i q_i^2 = p$.

Therefore, for $0 < \lambda \leqslant 1$, the polynomial M cannot be written as a finite sum of squares of C^3 functions and L cannot be written as a finite sum of squares of C^2 functions.

(b),(d) We write the proof of (b) using the polynomial M; the proof of (d) is again the same, but using L.

Let $\varphi(t) = \mathrm{e}^{-1/t^2}$ for $t \neq 0$ and $\varphi(0) = 0$. We take $f(x, y, z, t) = \varphi(t)M(x, y, z) + \psi(t)$ for a suitable nonnegative function $\psi : \mathbb{R} \to \mathbb{R}$ vanishing only at 0 that will be precised below. On its count, f vanishes only for $t = 0$. Let B be a ball centered at 0 in $\mathbb{R}^3$. We need the following easy lemma.

Lemma 1.3. *There are positive decreasing functions $C_\nu(\varepsilon)$ with the property that $\lim_{\varepsilon\to 0} C_\nu(\varepsilon) = +\infty$ and that for every decomposition $M + \varepsilon = \sum_1^\nu g_{j\varepsilon}^2$ with $g_{j\varepsilon} \in C^{3,\omega}(B)$ we have $\sum_1^\nu \|g_{j\varepsilon}\|_{C^{3,\omega}(B)} \geqslant C_\nu(\varepsilon)$.*

Proof. Assume the contrary: for arbitrarily small ε it would be possible to find decompositions of $M + \varepsilon$ in sums of squares with the $C^{3,\omega}$ norms of the $g_{j\varepsilon}$'s uniformly bounded and therefore with the $g_{j\varepsilon}$'s in a compact set of C^3. But then a suitable subsequence of them would converge to a decomposition of M in sums of squares of C^3 functions in B, which is impossible. $\square$

Now, a simple construction provides us with a decreasing function $C(\varepsilon)$ such that $\lim_{\varepsilon\to 0} C(\varepsilon) = +\infty$ and $\lim_{\varepsilon\to 0} C_\nu(\varepsilon)/C(\varepsilon) = +\infty$ for every ν. It suffices to choose a decreasing sequence (ε_n) such that $C_\nu(\varepsilon) \geqslant n^2$ for $\varepsilon \leqslant \varepsilon_n$ and $\nu \leqslant n$, and then to set $C(\varepsilon) = n$ for $\varepsilon_{n+1} \leqslant \varepsilon < \varepsilon_n$.

It is clearly possible to choose an increasing nonnegative function $\widetilde{\psi}(t)$, vanishing only at 0, such that $\widetilde{\psi}(t) = \mathrm{o}(t^N)$ for all N and that

$$\frac{1}{\varphi(t)^{1/2}} \leqslant C\left(\frac{\widetilde{\psi}(t)}{\varphi(t)}\right). \tag{1.2}$$

Set

$$\psi(t) = \int_{t/2}^{t} \widetilde{\psi}(s) h\left(\frac{t-s}{t}\right)\frac{ds}{t},$$

where h is a nonnegative C^∞ function with support in $(0, 1/2)$ and integral 1. Since $\widetilde{\psi}(t)$ is increasing, $\psi(t) \leqslant \widetilde{\psi}(t)$; but $C(\varepsilon)$ is decreasing, so the function ψ satisfies the same estimate (1.2) as $\widetilde{\psi}$ and belongs to C^∞. Now, if $f = \sum_{j=1}^\nu G_j^2$ with $G_j \in C^{3,\omega}$,

$$M(x, y, z) + \frac{\psi(t)}{\varphi(t)} = \frac{1}{\varphi(t)} \sum_{j=1}^{\nu} G_j^2(t, x, y, z).$$

But the $C^{3,\omega}(B)$ norm of the $G_j(t, \cdot)$ as t varies is bounded, which leads to a contradiction with (1.2). $\square$

Remark 1.4. As explained above, the nonnegative function f of Theorem 1.2 can be chosen strictly positive outside zero in cases (a) and (c). Whether this is possible also in cases (b) and (d), we do not know.

2. Admissible square roots

In [6] there is a well-known example of a C^∞ function whose square root is not C^2. A very similar function can be taken to show that it is possible that no admissible square root be $C^{1,\alpha}$ for any $\alpha \in (0, 1)$: namely, we can set

$$f(t) = \begin{cases} e^{-1/|t|}(\sin^2(\pi/|t|) + e^{-1/t^2}) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

More generally, the smaller are the minima of the oscillations near 0, the less regular are the admissible square roots; this leads us to the following generalization.

Theorem 2.1. *Given a modulus of continuity ω there exists a nonnegative function $f \in C^\infty(\mathbb{R})$ that vanishes only at* 0, *flat at* 0, *such that $h = (\sqrt{f})'$ is not ω-continuous on $\mathbb{R}$ (and therefore f has no $C^{1,\omega}$ admissible square root).*

Proof. Choose a function $\chi \in C^\infty(\mathbb{R})$ vanishing outside $(-2, 2)$, positive on $(-2, 2)$ and such that $\chi(t) = 1$ for $-1 \leqslant t \leqslant 1$. Let for $n \geqslant 1$

$$\rho_n = \frac{1}{n^2}, \quad t_n = 2\rho_n + \sum_{j=n+1}^{\infty} 3\rho_j,$$

$$I_n = [t_n - \rho_n, t_n + \rho_n], \quad J_n = [t_{n+1} + \rho_{n+1}, t_n - \rho_n],$$

$$\alpha_n = \frac{1}{2^n}, \quad \varepsilon_n = \omega^{-1}(\alpha_n/2) \quad \text{and} \quad \beta_n = \alpha_n \varepsilon_n^2$$

where ω^{-1} is the inverse function of ω. Note that by our hypotheses, $\varepsilon_n \leqslant \alpha_n/2 \leqslant \rho_n$ and then $t_n + \varepsilon_n \in I_n$. The function

$$f(t) = \begin{cases} \chi^2\left(-2^{\frac{t_1+2\rho_1-t}{\rho_1}}\right) + \sum_{n=1}^{\infty} \chi^2\left(\dfrac{t-t_n}{\rho_n}\right)[\alpha_n(t-t_n)^2 + \beta_n] & \text{if } t \geqslant 0, \\ f(-t) & \text{if } t < 0 \end{cases}$$

belongs to $C^\infty(\mathbb{R})$ and is strictly positive for $t \neq 0$, but h is not ω-continuous e.g. on $[-1, 1]$. Indeed, it is easy to obtain the estimate, for $t \in J_n \cup I_n$,

$$|f^{(k)}(t)| \leqslant C_k \alpha_n \rho_n^{-k} \xrightarrow[n\to\infty]{} 0,$$

while

$$\frac{|h(t_n + \varepsilon_n) - h(t_n)|}{\omega(\varepsilon_n)} = \frac{\alpha_n \varepsilon_n}{(\alpha_n \varepsilon_n^2 + \beta_n)^{1/2}\omega(\varepsilon_n)} = \frac{\sqrt{\alpha_n}}{\sqrt{2}\omega(\varepsilon_n)} = \frac{\sqrt{2}}{\sqrt{\alpha_n}}$$

that goes to infinity as $n \to \infty$. $\square$

Remark 2.2. Although the second derivative of $f^{1/2}$ is not bounded near 0, it is not difficult to see that $f^{1/2}$ is twice differentiable at that point (as in every other point). Indeed, a theorem in [1] ensures that if f is in $C^4(\mathbb{R})$, f has an admissible square root g such that $g''(x)$ exists at each point.

The set of points where g'' is continuous contains a nonempty open set, but it can have arbitrarily small measure (say in $[0, 1]$). Actually, let $K \subset [0, 1]$ be a Cantor-like compact set whose measure is $\geqslant 1 - \varepsilon$, and let us denote by I_n the connected components of its complement. It is not difficult, using the construction above, to find a nonnegative C^∞ function f_n, supported in I_n, such that $\|f_n\|_{C^n} \leqslant 2^{-n}$ and that g_n'' is unbounded for any admissible square root g_n of f_n. It is thus clear that $f = \sum_n f_n$ belongs to C^∞ and that g'' is unbounded near each point of K for any admissible square root g of f.

3. Admissible square roots of functions with controlled minima

It is a remark made by Glaeser in [6] that the points that most influence the behaviour of the first derivative of the (admissible) square root are the nonzero minima of f. In fact, we have

Theorem 3.1. *Let f be a nonnegative C^4 function of one variable such that it takes the value 0 at all its minima. Then f has an admissible square root in $C^2(\mathbb{R})$.*

Proof. Let F be the set of points x where f is flat, i.e. such that $f^{(k)}(x) = 0$ for $0 \leqslant k \leqslant 4$. The result being easy if $F = \emptyset$, we may assume that $0 \in F$. Let A_i be the connected components of $\mathbb{R} \setminus F$. In each interval A_i, the points where f vanishes cannot have an accumulation point in A_i and they can be shared out amongst two sequences indexed by $\mathbb{Z}$ or an interval of $\mathbb{Z}$

- the points $\ldots z_{i,\nu} < z_{i,\nu+1} \ldots$ such that $f(z_{i,\nu}) = 0$ and $f''(z_{i,\nu}) > 0$,
- the points $\ldots z'_{i,\kappa} < z'_{i,\kappa+1} \ldots$ such that $f(z'_{i,\kappa}) = f''(z'_{i,\kappa}) = 0$ whereas $f^{(4)}(z'_{i,\kappa}) > 0$.

In each interval A_i, let us fix a function g_i such that:

- g_i is continuous and $g_i(x)^2 = f(x)$ for $x \in A_i$,
- the sign of $g_i(x)$ changes when x crosses the $z_{i,\nu}$,
- the sign of $g_i(x)$ does not change when x crosses the $z'_{i,\kappa}$.

The function g_i is uniquely determined up to its sign and belongs to $C^2(A_i)$, which is a classical consequence of the Taylor expansion. The function g will be defined on $\mathbb{R}$ by $g(x) = g_i(x)$ for $x \in A_i$ and by $g(x) = 0$ for $x \in F$.

Let us denote by $d(x)$ the distance of x to F. The main part of the proof is contained in the following lemma.

Lemma 3.2. *For any $R > 0$, there exists a continuous nonnegative function β defined on $[-R, R]$ such that $\beta^{-1}(0) = F \cap [-R, R]$ and that*

$$\left|g_i^{(k)}(x)\right| \leqslant d(x)^{2-k}\beta(x) \quad \textit{for } 0 \leqslant k \leqslant 2 \textit{ and } x \in A_i \cap [-R, R]. \tag{3.1}$$

Assuming the lemma true, given $x \in \mathbb{R}$ and choosing $R > |x|$, it is clear that the inequalities (3.1) imply that g is of class C^2 in a neighbourhood of x. In fact, if $x \in \cup_i A_i = \mathbb{R} \setminus F$ we already know it, while if $x \in F$ we prove easily using the lemma that the limits of g, g' and g'' at x exist and are 0.

This concludes the proof of Theorem 3.1; we now pass to the proof of Lemma 3.2.

The function β can be taken equal to $C\alpha^{1/2}$ (or to any larger continuous function vanishing on F) where the function α is defined as follows. Let ω be a modulus of continuity for the restriction of $f^{(4)}$ to $[-2R, 2R]$, defined on $[0, 4R]$. Setting $\alpha(x) = \omega(d(x))$, one has

$$\left| f^{(k)}(x) \right| \leqslant d(x)^{4-k} \alpha(x) \quad \text{for } 0 \leqslant k \leqslant 4 \text{ and } |x| \leqslant R. \tag{3.2}$$

Actually, for $x \in A_i \cap [-R, R]$, one has $d(x) = |x - y|$ with $y \in F \cap \overline{A_i} \cap [-2R, 2R]$ and the estimates follow by integration (here we use that $0 \in F$). Thanks to the concavity of ω, one has also $1/2 \leqslant \alpha(z)/\alpha(x) \leqslant 2$ for $|z - x| \leqslant d(x)/2$.

We already know that $g_i \in C^2$ and it is thus sufficient to prove the estimates (3.1) when $f(x) > 0$.

Set

$$\rho(x) = \max\left(\left(\frac{f(x)}{\alpha(x)} \right)^{1/4} ; \left(\frac{[f''(x)]^+}{\alpha(x)} \right)^{1/2} \right).$$

In view of (3.2) one has $\rho(x) \leqslant d(x)$. We can thus apply to the function $\varphi(t) = \alpha(x)^{-1}\rho(x)^{-4} f(x + t\rho(x))$, defined on $[-1/2, 1/2]$, the following lemma, which is the key of the proof of the Fefferman–Phong inequality (see Hörmander [9, Lemma 18.6.9] for the proof, although his statement is slightly different).

Lemma 3.3. *Let φ be a nonnegative C^4 function defined on $[-\frac{1}{2}, \frac{1}{2}]$ such that* $\max(\varphi(0), \varphi''(0)) = 1$ *and that* $\sup_{|t| \leqslant 1/2} \left| \varphi^{(4)}(t) \right| \leqslant 2$. *There exist universal constants $C_0 \geqslant 1$ and $r \in (0, 1/2)$ such that*

$$\left| \varphi^{(k)}(t) \right| \leqslant C_0 \quad \textit{for } 0 \leqslant k \leqslant 4 \textit{ and } |t| \leqslant 1/2. \tag{3.3}$$

If, moreover, one has $\varphi(0) \leqslant C_0^{-1}$, then $\varphi''(t) > 1/2$ for $|t| \leqslant r$ and there exists $s \in (-r, r)$ such that $\varphi(s) = \min_{|t| \leqslant r} \varphi(t)$.

Let us consider the two following cases.

(1) One has $f(x)/\alpha(x) \geqslant C_0^{-4} \rho(x)^4$. Thanks to (3.3), we know that $\left| f^{(k)}(x) \right| \leqslant C_0 \alpha(x) \rho(x)^{4-k}$ and it is easy to estimate the first and second derivatives of $g_i = \pm f^{1/2}$ at x. One has

$$\frac{\left| f'(x) \right|}{f(x)^{1/2}} \leqslant C_0^3 \alpha(x)^{1/2} \rho(x); \quad \frac{\left| f''(x) \right|}{f(x)^{1/2}} \leqslant C_0^3 \alpha(x)^{1/2}; \quad \frac{f'(x)^2}{f(x)^{3/2}} \leqslant C_0^8 \alpha(x)^{1/2}.$$

The estimates (3.1) are thus proved in this case, if only $\beta(x) \geqslant 2C_0^8 \alpha(x)^{1/2}$.

(2) One has $f(x)/\alpha(x) \leqslant C_0^{-4}\rho(x)^4$. We know that f restricted to $I_x = (x - r\rho(x), x + r\rho(x))$ has a minimum at some point $y \in I_x$ and the assumption of Theorem 3.1 says that $f(y) = 0$. Moreover, by Lemma 3.3, we have $\left|f^{(k)}(z)\right| \leqslant C_0\alpha(x)\rho(x)^{4-k}$ and $f''(z) \geqslant \alpha(x)\rho(x)^2/2$ for $z \in I_x$. By the Taylor expansion, we have

$$\pm g_i(z) = (z - y)\left(\int_0^1 (1 - s) f''\big((1-s)y + sz\big)\, ds\right)^{1/2} = (z - y)H(z)^{1/2}.$$

One has $H(z) \geqslant \alpha(x)\rho(x)^2/4$, while $\left|H'(z)\right| \leqslant C_0\alpha(x)\rho(x)$ and $\left|H''(z)\right| \leqslant C_0\alpha(x)$. It is then easy to estimate the derivatives at the point $z = x$ of the function $z \to (z-y)H(z)^{1/2}$. One gets, with a universal constant, $\left|g_i^{(k)}(x)\right| \leqslant C_1\alpha(x)^{1/2}\rho(x)^{2-k}$ for $k = 0, 1, 2$.

The proof of (3.1), and thus of Theorem 3.1, is complete, choosing $\beta(x) = (C_1 + 2C_0^8)\alpha(x)^{1/2}$. $\square$

Remark 3.4. Under the assumptions of Theorem 3.1, if moreover $f(y) = 0 \Rightarrow f''(y) = 0$ (i.e. there are no points $z_{i,\nu}$), the proof above shows that $f^{1/2}$ belongs to C^2.

Actually, the obstacle to the existence of a C^2 admissible square root for a nonnegative C^4 function comes from the converging sequences of "relatively small" nonzero minima. One has indeed the following modification of Theorem 3.1.

Theorem 3.5. *Let f be a nonnegative C^4 function on $\mathbb{R}$; f has a C^2 admissible square root if and only if there exists a continuous function γ vanishing on F such that, for any minimum x_0 of f where $f(x_0) > 0$, $f''(x_0) \leqslant \gamma(x_0) f(x_0)^{1/2}$.*

Proof. The condition in the theorem is equivalent to the following: for any sequence x_n of nonzero minima of f which converges towards a point of F, one has $f''(x_n)/f(x_n)^{1/2} \to 0$. We repeat the proof of Theorem 3.1, keeping the same function α and thus the same function ρ, but we will have to enlarge the function β. What is changed is that, in case 2, we also have to consider the possibility that at the minimum point $y \in I_x$ we have $f(y) > 0$ (but then, by our hypothesis, also $f(y)^{1/2} \geqslant f''(y)/\gamma(y)$). Define

$$\Gamma(x) = \sup_{z \in I_x} \gamma(z),$$

Γ is again continuous and vanishing on F, since $\rho(x) < d(x)$.

Now, for $\xi \in I_x$, by Lemma 3.3

$$\tfrac{1}{2} f''(x) \leqslant f''(\xi) \leqslant C_0\alpha(x)\rho(x)^2 = C_0 f''(x);$$

and thus

$$\left|\frac{f'(x)}{2f(x)^{1/2}}\right| \leqslant \frac{\big(\sup_{I_x} f''(\xi)\big)|x - y|}{2f(y)^{1/2}} \leqslant \frac{2C_0 f''(y)|x - y|}{2f(y)^{1/2}}$$
$$\leqslant C_0\gamma(y)|x - y| \leqslant C_0\Gamma(x)\rho(x).$$

At the same time,

$$\frac{f''(x)}{2f(x)^{1/2}} \leqslant \frac{2f''(y)}{2f(y)^{1/2}} \leqslant \gamma(y) \leqslant \Gamma(x),$$

while

$$f(x) \geqslant f(y) + \frac{1}{4C_0} f''(y)(x-y)^2$$

therefore, for the second term in $g''(x)$ one has

$$\frac{f'(x)^2}{4f(x)^{3/2}} \leqslant \frac{4C_0^2 f''(y)^2(x-y)^2}{4f(y)^{1/2}\frac{1}{4C_0}f''(y)(x-y)^2} \leqslant 4C_0^3\gamma(y) \leqslant 4C_0^3\Gamma(x).$$

It is then sufficient to choose $\beta(x)$ also larger than $(4C_0^3+1)\Gamma(x)$ to obtain the inequalities 3.1. The proof is complete.

Conversely, let us assume that f has a C^2 admissible square root g, but there is a sequence x_n of nonzero minima of f converging towards a point $\bar{x} \in F$ with

$$\lim_{n\to\infty} \frac{f''(x_n)}{f(x_n)^{1/2}} > 0.$$

Then, since $f'(x_n) = 0$ for every n,

$$g''(\bar{x}) = \lim_{n\to\infty} g''(x_n) > 0$$

which is impossible, since the first 4 derivatives of f vanish at $\bar{x}$ by definition of F. $\square$

It is clear that the regularity assumption of Theorem 3.1 cannot be weakened to $f \in C^{3,1}$ (take $f(x) = x^4 + \frac{1}{2}x^3|x|$). The following theorem says that the conclusion cannot be improved either, not even starting with a C^∞ function.

Theorem 3.6. *For any given modulus of continuity ω there is a C^∞ nonnegative function f on $\mathbb{R}$, taking the value* 0 *at all its minima, which has no $C^{2,\omega}$ admissible square root.*

Proof. Let $\chi \in C^\infty(\mathbb{R})$ be the even function with support in $[-2,2]$ defined by $\chi(t) = 1$ for $t \in [0,1]$ and by $\chi(t) = \exp\left\{\frac{1}{(t-2)e^{1/(t-1)}}\right\}$ for $t \in (1,2)$. We note that the logarithm of χ is a concave function on $(1,2)$. For every $(a,b) \in [0,1] \times [0,1]$, $(a,b) \neq (0,0)$, the function $t \mapsto \log(at^4 + bt^2)$ is concave on $(0,+\infty)$ and thus the function

$$t \mapsto \chi^2(t)(at^4 + bt^2)$$

has only one local maximum and no local minimum in $(1,2)$ (its logarithmic derivative vanishes exactly once). Set

$$\rho_n = \frac{1}{n^2}, \quad t_n = 2\rho_n + \sum_{j=n+1}^{\infty} 5\rho_j,$$

$$I_n = [t_n - 2\rho_n, t_n + 2\rho_n], \quad \alpha_n = \frac{1}{2^n},$$

$$\varepsilon_n = \omega^{-1}(\alpha_n/2), \quad \beta_n = \alpha_n \varepsilon_n^2.$$

Note that the I_n's are closed and disjoint and that one has $\varepsilon_n \leqslant \alpha_n/2 \leqslant \rho_n$ as in Theorem 2.1. Define

$$f(t) = \sum_{n=1}^{\infty} \chi^2 \left(\frac{t - t_n}{\rho_n} \right) (\alpha_n (t - t_n)^4 + \beta_n (t - t_n)^2).$$

Indeed, $f \in C^\infty(\mathbb{R})$: this is clear except perhaps at the origin where it is sufficient to note that for $t \in I_n$ (where we can also estimate $t - t_n$ with ρ_n)

$$|f^{(k)}(t)| \leqslant C_k \rho_n^{2-k} \alpha_n \xrightarrow[n \to \infty]{} 0.$$

Moreover, f takes the value 0 at all its local minima which are the points t_n and the points between $t_{n+1} + 2\rho_{n+1}$ and $t_n - 2\rho_n$. On the other hand, in a fixed interval I_n, f admits only two C^1 roots, namely

$$\pm \chi \left(\frac{t - t_n}{\rho_n} \right) (t - t_n) \sqrt{\beta_n + \alpha_n (t - t_n)^2},$$

therefore, any C^1 admissible square root of f is of the form

$$g(t) = \sum_{n=1}^{\infty} \sigma_n \chi \left(\frac{t - t_n}{\rho_n} \right) (t - t_n) \sqrt{\beta_n + \alpha_n (t - t_n)^2}$$

for some choice of the signs $\sigma_n = \pm 1$. Observing that $\chi^{(k)}(0) = \chi^{(k)}(\varepsilon_n/\rho_n) = 0$ for all $k > 0$, we get

$$\begin{aligned} \frac{|g''(t_n + \varepsilon_n) - g''(t_n)|}{\omega(\varepsilon_n)} &= \frac{|g''(t_n + \varepsilon_n)|}{\omega(\varepsilon_n)} \\ &= \frac{2\alpha_n \varepsilon_n}{\omega(\varepsilon_n)(\beta_n + \alpha_n \varepsilon_n^2)^{1/2}} + \varepsilon_n \frac{\alpha_n(\beta_n + \alpha_n \varepsilon_n^2) - \alpha_n^2 \varepsilon_n^2}{\omega(\varepsilon_n)(\beta_n + \alpha_n \varepsilon_n^2)^{3/2}} \\ &= \frac{3\alpha_n \varepsilon_n \beta_n + 2\alpha_n^2 \varepsilon_n^3}{\omega(\varepsilon_n)(\beta_n + \alpha_n \varepsilon_n^2)^{3/2}} = \frac{5\sqrt{\alpha_n}}{\sqrt{8}\omega(\varepsilon_n)} = \frac{5}{\sqrt{2\alpha_n}} \end{aligned}$$

that goes to infinity as $n \to \infty$. $\square$

Acknowledgements

We would like to thank F. Acquistapace and N. Lerner for their useful suggestions.

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